Students' Use of Second-Order Relationships in Algebra

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The notion of a variable appears critical to the understanding of school algebra. Research studies are generally in agreement that this concept requires higher- order (i.e., formal) cognitive skills. A feature associated with the understanding of a variable involves the use of second-order relationships, that is, where the letters are themselves relationships. The purpose of this investigation was to explore this feature and to extend the findings identified earlier by Pegg and Coady (1993). The SOLO Taxonomy was used to interpret students' responses.

Introduction

Attempts at identifying the characteristics of higher-order algebraic thinking have been made by Pegg and Coady (1993) and Coady and Pegg (in press). This was carried out by building on the previous research efforts of Collis (1975) and Küchemann (1981). In this work the authors were able to show the value of the SOLO Taxonomy (see Biggs and Collis (1982, 1991) and Pegg (1992) for more details concerning the SOLO Taxonomy) in classifying students' responses to questions which involved the notion of a variable. In addition to identifying levels of growth on a series of questions, the authors were able to give additional meaning to the two relevant modes of functioning, namely, Concrete Symbolic and Formal.

Concrete Symbolic responses are closely linked with real world experiences and observations. In the case of algebra, this is evident by the manipulation of symbols. In this mode, students are able to simplify expressions, expand brackets, solve simple equations and substitute numerical values for letters. However, responses are characterised by relatively quick closures and there is little indication of the recognition of constraints imposed by the mathematical system in which the students are functioning. A lack of consideration of alternatives is also a feature of the responses within this mode.

Formal responses indicate the ability to extract abstract concepts from generalisations. Operations on these concepts can then be carried out without the need for a concrete referent. In the case of algebra, this is evident by an ability to consider possible limitations or constraints inherent in the mathematical structure of the system that are not overtly present. The relationships between variables in a question become the focus of attention, rather than the obvious features related to the variable itself. Overriding generalisations between the variables are identified and used explicitly.

The purpose of this current investigation is to extend these findings to the case of variable substitution. That is, in the words of Küchemann (1981), to explore the notion of 'second-order relationships', wherein the "elements are themselves relationships" (p.111). The remainder of this paper reports empirical evidence which clearly distinguishes the difference between Concrete Symbolic responses and early Formal responses within the theme of variable substitution. It also provides student examples that can be interpreted within an early unistructural-multistructural-relational learning cycle in the Formal mode.

Methodology

In order to address the research issue mentioned above, a series of questions requiring the use of variable substitution was given to 147 first-year university students. These students were aged between 17 and 20 years and were all enrolled in mathematically-based applied science degree courses. These students were drawn from the top 50% of the age cohort and approximately 80% of the group had undertaken a calculus-based mathematics course in their final two years of secondary school. Three questions are discussed below. These have been chosen as they are representative of the questions asked.

Results and Discussion

Question 1: If \f(a,b) = 4, find the value of $\frac{a+b}{a-b}$

Most of the responses to this question fell into two distinct categories. The first group focussed on numerical substitution and as such were coded as representative of the Concrete Symbolic mode. The second group were able to work with variable substitution, albeit with varying degrees of success. Such responses were coded within levels of the Formal mode. In addition, a small group of students attempted algebraic manipulation but made no attempt to use variable substitution. This group appeared transitional in nature between the two previous groups.

Concrete Symbolic Responses

Within this mode, students made numerical substitutions for *a* and *b*, based on the relationship $\frac{a}{b} = 4$. For example:

$$\frac{a}{b} = 4 \quad a = 4 \quad b = 1$$

$$\therefore \frac{a+b}{a-b} = \frac{4+1}{4-1} = \frac{5}{3}$$

This use of concrete referents clearly delineates this group of responses from those classified as Formal, where the more abstract procedure of variable substitution was used.

Formal Responses

Two distinct solution strategies were identified from the analysis of the written scripts. One group chose to work with the fraction $\frac{a+b}{a-b}$ while the second chose to focus their attention on the equation $\frac{a}{b} = 4$. In each group, examples of an early unistructural-multistructural-relational learning cycle were discernible.

<u>Group 1</u>: *Unistructural responses*: Manipulation of the expression $\frac{a+b}{a-b}$ took place in order to make use of $\frac{a}{b} = 4$. The key feature of this type of response was that students were aware of the need to generate $\frac{a}{b}$ from $\frac{a+b}{a-b}$ so that the substitution $\frac{a}{b} = 4$ could be made. This match was carried out at the expense of correct manipulative procedures. For example:

$$1.\ \frac{a+b}{a-b} = a + b \div a - b$$

$$= \frac{a}{-b} + \frac{b}{a} = -4 + \frac{b}{a}$$

$$2\ \ \frac{a+b}{a-b} = \frac{a}{a-b} + \frac{b}{a-b}$$

$$= \frac{-4}{a} + \frac{1}{4-b}$$

Multistructural responses: Responses in this category indicated that students again needed to rearrange $\frac{a+b}{a-b}$ into a useful form so that $\frac{a}{b} = 4$ could be utilised. The factor distinguishing these responses from those coded unistructural was the improvement shown in the ability to correctly monitor the symbolic manipulation techniques used. However, students failed to reach the conclusion as they lacked the ability to have an overview of the entire set of procedures required. For example:

$$\frac{a+b}{a-b} = \frac{a(1+\frac{b}{a})}{b(\frac{a}{b}-1)}$$

if $\frac{a}{b} = 4$ then

$$\frac{a(1+\frac{b}{a})}{b(\frac{a}{b}-1)} = 4\frac{(1+\frac{b}{a})}{(\frac{a}{b}-1)}$$

Relational responses: Responses in this category demonstrated that students had an overview of the demands of the question and could utilise appropriate manipulative skills to achieve the correct result. For example:

$$\frac{a+b}{a-b} = \frac{\frac{a}{b}+1}{\frac{a}{b}-1} = \frac{4+1}{4-1} = \frac{5}{3}$$

<u>Group 2:</u> *Unistructural responses*: These responses showed that students were prepared, on their own initiative, to rearrange the given conditions to better suit the demands of the question, but once this had taken place these students were at a loss as to how to proceed. For example:

$$a = 4b \quad b = \frac{a}{4}$$

Multistructural responses: Responses of this type also began with some transformation of $\frac{a}{b} = 4$ taking place. This was followed by variable substitution, but a clear overview of the question was not in evidence as substitutions were made for both *a* and *b*. Once these substitutions had taken place, the result was often more complex in structure than the original question. Students did not make any attempt to address this anomaly. For example:

1. $a = 4b \quad b = \frac{a}{4}$

$$\therefore \frac{a+b}{a-b} = \frac{4b+\frac{a}{4}}{4b-\frac{a}{4}}$$

2. $a = 4b \quad b = \frac{a}{4}$

$$\frac{4b+\frac{a}{4}}{4b-\frac{a}{4}} = \frac{16b+a}{4} \times \frac{4}{16b-a}$$

Relational responses: A response at this level indicated that a strategy which had considered all aspects of the question was in place, with students having sufficient control over the necessary manipulative skills to reach the correct conclusion. For example:

1. $a = 4b$

$$\frac{a+b}{a-b} = \frac{4b+b}{4b-b} = \frac{5}{3}$$

2. $$\frac{a+b}{a-b} = \frac{a}{a-b} + \frac{b}{a-b} = \frac{4b}{4b-b} + \frac{b}{4b-b} = \frac{4b}{3b} + \frac{b}{3b} = \frac{5}{3}$$

Transitional Responses

The study also revealed the existence of a third group of responses that utilised reasonably sophisticated manipulative techniques involving the rearrangement of variable relationships. This, arguably could have formed the first step in solving the problem, but as the result of this did not immediately yield anything that could be utilised in terms of the information given in the question, further processing was discontinued. For example:

$$\frac{a+b}{a-b} \times \frac{a+b}{a+b} = \frac{a^2+2ab+b^2}{a^2-b^2}$$

Question 2: If $p = 2q$ and $q = st$ find pq in terms of t, given that $s = \frac{1}{2}$

Two clear groups of responses reflecting Concrete Symbolic and Formal modes were identified.

Concrete Symbolic Responses

Responses in this category were confined to students making the substitution $s = \frac{1}{2}$. There was no evidence of the students' attention being directed at p or pq. For example:

$q = st \quad = \frac{1}{2}t$

Formal Responses

Unistructural responses: Responses here focussed on one variable substitution. Students worked with either p or q, but once an expression had been found, this was not used to find pq.

1. $p = 2q = 2st = t$
2. $q = st = \frac{1}{2}t = 2 \times \frac{1}{2}t = 2 \times \frac{1}{2}t = t$

Multistructural responses: This level of response was indicated when students chose to work with pq from the outset, but became lost in the symbolism. As a result students did not appear to have an overview of the question.

1 $pq = 2q\,st = 2q \times \frac{1}{2}t$

$\therefore pq = qt$

2 $pq = 2qst = 2q^2$

Relational responses: Responses here showed that an integration of all data had taken place with the end product always in sight. Two methods were used at this level. Students initially worked with either p only or with pq.

1. $p = 2q = 2st = 2 \times \frac{1}{2}t = t$

2. $pq = 2q\,st = 2stst$

$= 2s^2t^2 = 2 \times \frac{1}{4}t^2$

$\therefore pq = stt = \frac{1}{2}t^2 = \frac{1}{2}t^2$

Question 3: Express $3a - b + 4c$ in terms of b, given that $a + 1 = b = c - 1$

The responses showed similar trends to those identified in **Question 1** and **2** above. Also there was evidence of a transitional group similar to that identified in **Question 1.**

Concrete Symbolic Responses

A response at this level indicated that the students needed to work within an 'equation' framework, such as:

$3a - b + 4c = 0$

$b = 3a + 4c$

There was no attempt to utilise variable substitution or address the additional information provided.

Formal Responses

Students using this mode of reasoning were able to deconstruct the given information. However it was only at the relational level that students were able to see the value of rearranging the conditions in order to express the answer in terms of a single letter.

Unistructural responses: The responses at this level showed students were capable of making one substitution only. Once this was completed, students had invariably eliminated b altogether from the expression $3a - b + 4c$, as the following examples show.

1 $3a - b + 4c = 3a - (a + 1) + 4c$

2 $3a - b + 4c = 3a - (c - 1) + 4c$

Multistructural responses: These responses were characterised by the independent substitutions of $b = a + 1$ and $b = c - 1$ leading to two 'answers'. Students appeared to have lost sight of the question and as with the previous level of response, b was removed. While these substitutions were made in sequence there was no evidence that the students had an overall plan.

$b = a + 1$	$b = c - 1$
$[3a-b+4c] = 3a-(a+1)+4c$	$= 3a-c+1+4c$
$[3a-b+4c] = 3a-1(c-1)+4c$	$= 3a+3c+1$
$= 3a-a-1+4c$	
$= 2a-1+4c$	

Relational responses: Manipulative expertise in conjunction with an immediate grasp of the requirements of the question resulted in the successful completion of this question.

$a + 1 = b = c - 1 \quad \therefore b = a + 1$ and $b = c - 1$

$b - 1 = a \quad b + 1 = c$

$\therefore 3a - b + 4c = 3(b - 1) - b + 4(b + 1)$

$= 3b - 3 - b + 4b + 4$

$= 6b + 1$

Transitional Responses

Responses in this group went further than using equation solving by attempting to incorporate the information given. However, processing remained locked into the Concrete Symbolic mode of functioning, in this case, equation solving. For example:

1 $b = 3a + 4c \quad b = 3a + 4c$

$a + 1 = 3a + 4c$ when $b = a + 1$

$c - 1 = 3a + 4c$ when $b = c - 1$

$= 2a - 1 + 4c \quad = 3a + 1 + 3c$

2. If $a + 1 = c - 1 \quad b = a - c + 2 = 0$

$\therefore 3a - (a - c + 2) + 4c = b$

$3a - a + c - 2 + 4c = b$

$b = 2a + 5c - 2$

Implications and Conclusion

The general theme of this paper was the examination and classification of students' responses to questions requiring variable substitution. The process was assisted by the responses being interpreted within the constructs of the SOLO Taxonomy. To be successful, students had to be prepared to substitute letters which were themselves relationships and undertake correct manipulative procedures. Generally, the test results were not encouraging, with only 39%, 23% and 31% of students correct (providing a relational response in the Formal mode) on **Questions 1, 2** and **3** respectively. This reflects poorly on the students' ability to operate on variables as entities in their own right and suggests that the second-order relationships that are involved in the notion of a variable are inadequately recognised, let alone understood by many of the students in the sample.

Despite the differences in the question formats, there was a degree of consistency in student responses. For example, in **Questions 1** and **2**, responses in the Concrete Symbolic mode implied numerical substitution of some form, either initiated by the students, as in the case of **Question 1**, or adopted as part of the information provided, as in **Question 2**. In **Questions 1** and **3**, transitional responses were noted. These appeared to have two characteristics, namely, they included symbolic manipulation but ignored variable substitution or they included some form of variable substitution but within an incorrect context.

The findings also parallel those found earlier by the authors in that qualitatively different responses can be identified which clearly demonstrate Concrete Symbolic and early Formal thinking. In the former case, competent manipulative skills in the form of numeric substitution or equation solving techniques were manifest, but no use was made of second-order relationships. In the latter case, responses showed a gradual improvement from unistructural to relational levels in the use of second-order relationships. The responses were marked by appropriate manipulative procedures and an increased ability to control all the elements in the question.

This study has again confirmed the applicability of the SOLO Taxonomy as an assessment tool in gauging the functional performance of a student. It has also provided further insights into the nature of students' understanding of algebra. One aspect that deserves further consideration is associated with the transitional groups identified in **Questions 1** and **3**. This issue is part of a more global concern namely, what role does performance in the Concrete Symbolic mode play in supporting or hindering solution attempts of questions that require thinking associated with the Formal mode? It is clear that an answer to this question would greatly assist our knowledge of students' understanding of algebra and help place manipulative skills within a clearer context.

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